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The dynamics of a "flexible-rotor/limited-power-excitation-source" system☆

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ABSTRACT

A "flexible-rotor/limited-power-excitation-source" dynamical system, which is a model of the vibrations of shafts driven by a motor of limited power is considered. The types of rotation characteristics of the motor and resonance characteristics of the shaft in the resonance parameter zone for different values of the viscosity coefficient of the medium are investigated. These characteristics completely describe the dynamics of the system when its parameters vary.

The experimental investigation and use of rotating shafts (turbomachines for various purposes, centrifugal pumps, etc., which are called "flexible rotors") have shown that in certain situations the power of their flexural vibrations is comparable with the power of the actuating mechanism. This becomes possible as the shaft accelerates to the working rotation frequency and passes through its resonance frequencies, as well as during shutdown of the system. A situation in which the working frequency is fairly close to one of the resonance frequencies of the shaft can arise. Finally, resonance frequencies can drift to the working rotation frequency as a consequence of changes in the vibrational properties of the shaft itself during service. In these cases an investigation of the dynamics and a calculation of the parameters of shafts should be performed taking into account the dynamical properties of the actuating mechanism. An investigation of a coupled "flexible-rotor/actuating-mechanism" system is needed.

The investigation of vibrating systems with limited-power excitation sources begun by Sommerfeld[1] was subsequently continued (Refs 2–16, etc.). The phenomenon of the rotation frequency of the actuating system "becoming stuck" at a constant value in the vicinity of the resonance frequency of the vibrating system, which has been named the Sommerfeld effect,[4] also occurs in the case of rotating shafts. This effect also explains the build up of the vibrations of a shaft in the resonance parameter zone.

The difficulties in an analytical investigation of "flexible-rotor/actuating-mechanism" dynamical systems are attributed not only to their non-linearity and cylindrical phase space, but also to their high dimension. In particular, a system consisting of an unbalanced shaft interacting with a limited power source has three degrees of freedom in the single-mode approximation (Fig. 1). In such a case the method of averaging is actually the only analytical method for investigating the system dynamics. However, its use in the standard implementation (the use of amplitude–phase replacements, which bring the system into a standard form) leads to averaged systems whose investigation is a no less a problem than the investigation of the original systems.[5,10] The method is effective in the proposed form, in which the dimension of the system is reduced to half, and the averaged system is specified in three-dimensional phase space. The model considered is far from new.[5,10,17] Nevertheless, because of the importance of this model, particularly from the practical standpoint, we will return to it and investigate its dynamical properties in detail.

1. The model

We will assume that the mass distributed along the shaft length is reduced to the mass of a disk located at the middle of a weightless shaft that is not torsionally compliant. The disk eccentricity is denoted by e, W is the geometric centre of the disk, S is the centre of gravity, and O is the equilibrium position of the disk (Fig. 1).

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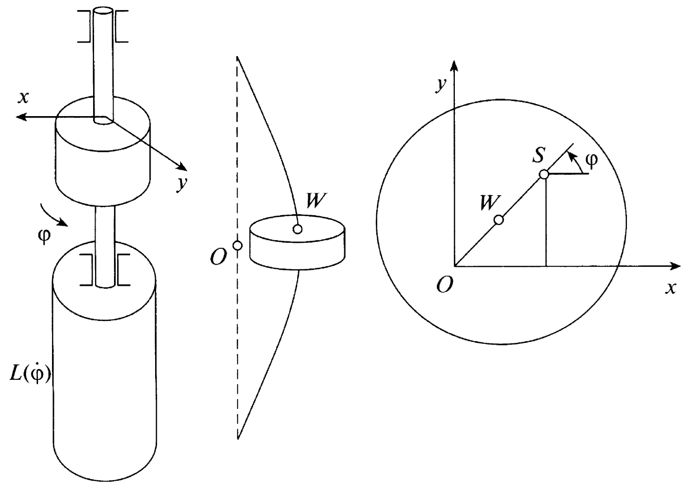

Fig. 1.

In dimensional variables and parameters the system dynamics is described by the equations[5,10]

$$m\ddot{x} + \varepsilon\dot{x} + cx + k(\dot{x} + \dot{\varphi}y) = ce\cos\varphi, \quad m\ddot{y} + \varepsilon\dot{y} + cy + k(\dot{y} - \dot{\varphi}x) = ce\sin\varphi$$
$$J\ddot{\varphi} = L(\dot{\varphi}) - q\dot{\varphi} - ce(x\sin\varphi - y\cos\varphi) - k(\dot{x}y - \dot{y}x) - k\dot{\varphi}\left(x^2 + y^2\right) \tag{1.1}$$

Here x and y are the coordinates of the centre of gravity of the disk in the fixed system of coordinates with origin on the axis of the unperturbed shaft and the (x, y) plane perpendicular to the axis, ε and k are the external and internal damping factors for flexural vibrations, c is the shaft stiffness at the point of attachment of the disk, m is the mass of the disk, J is the moment of inertia of the motor rotor, φ is the range of rotation, and $L(\dot{\varphi})$ is the torque of the motor, which includes the moment of the internal forces of resistance to the rotor motion. The torque of an asynchronous motor is known to be an essentially linear function of the form $L(\dot{\varphi}) = T - q\dot{\varphi}$, where T is a constant component and $q\dot{\varphi}$ is the moment of the forces of resistance to the rotor motion.[18] We will adhere to this assumption below.

In dimensionless variables, parameters and times, system (1.1) has the form

$$\ddot{x} + \frac{\varepsilon\omega_0}{c}\dot{x} + x + \frac{k\omega_0}{c}(\dot{x} + \dot{\varphi}y) = \frac{e}{A_0}\cos\varphi, \quad \ddot{y} + \frac{\varepsilon\omega_0}{c}\dot{y} + y + \frac{k\omega_0}{c}(\dot{y} - \dot{\varphi}x) = \frac{e}{A_0}\sin\varphi$$
$$\frac{J\omega_0^2}{T_0}\ddot{\varphi} = \frac{T}{T_0} - \frac{q\omega_0}{T_0}\dot{\varphi} - \frac{ceA_0}{T_0}(x\sin\varphi - y\cos\varphi) - \frac{k\omega_0A_0^2}{T_0}\left(\dot{x}y - \dot{y}x + \dot{\varphi}\left(x^2 + y^2\right)\right) \tag{1.2}$$

When we changed from system (1.1) to system (1.2), we introduced the dimensionless time

$$\omega_0 t = \tau, \quad \omega_0^2 = c/m$$

and we performed scaling of the variables x and y by the constant A_0 and scaling of the moments by the constant T_0. The values of the scale factors are of no fundamental significance. The former notation is retained for the dimensionless variables.

We will introduce the new parameters

$$\frac{c}{m} = \omega_0^2, \quad \left(\frac{J\omega_0^2}{T_0}\right)^{-1} = \mu, \quad \frac{\varepsilon\omega_0}{c} = \mu h, \quad \frac{k\omega_0}{c} = \mu h_1, \quad \frac{e}{A_0} = \mu\nu,$$
$$\frac{ceA_0}{T_0} = \mu\lambda, \quad \frac{k\omega_0A_0^2}{T_0} = \mu\chi, \quad \frac{T - q\omega_0}{T_0} = \mu\Delta$$

as well as the new variable $\dot{\varphi} = 1 + \mu\xi$. System (1.2) now takes the form

$$\dot{x} = x_1, \quad \dot{x}_1 = -x + \mu F_1, \quad \dot{y} = y_1, \quad \dot{y}_1 = -y + \mu F_2,$$
$$\dot{\xi} = \mu F_3, \quad \dot{\varphi} = 1 + \mu\xi \tag{1.3}$$

where

$$F_1 = \nu\cos\varphi - (h + h_1)x_1 - h_1 y, \quad F_2 = \nu\sin\varphi - (h + h_1)y_1 + h_1 x$$

$$F_3 = \Delta - \gamma\xi - \lambda(x\sin\varphi - y\cos\varphi) - \chi(x_1 y - y_1 x) - \chi\left(x^2 + y^2\right)$$

Terms of the order of μ^2 are not written out in system (1.3) since they do not have any effect on the first approximation of the averaged system.

If the parameter μ is sufficiently small, system (1.3) is quasilinear. Effective interaction of the vibrations in such systems is known to occur only in regions of parameter values that correspond to "full" resonances. Outside the resonance zones, the system dynamics is trivial.

The parameter ratio $(T-q\omega_0)/T_0=\mu\Delta$ corresponds to the zone of fundamental resonance:

$$\frac{T-q\omega_0}{T_0}=\frac{(\Omega_0-\omega_0)q}{T_0}=\mu\Delta$$

i.e., $\Omega_0\approx\omega_0$, where Ω_0 is the partial rotation frequency of the motor. The system dynamics in this parameter zone will be investigated below by the method of averaging.

In system (1.3) we make the replacement of variables

$$x=u_1\sin\varphi+\upsilon_1\cos\varphi,\quad x_1=u_1\cos\varphi-\upsilon_1\sin\varphi$$
$$y=u_2\sin\varphi+\upsilon_2\cos\varphi,\quad y_1=u_2\cos\varphi-\upsilon_2\sin\varphi$$

As a result, we obtain the following equivalent system in standard form with a rapidly rotating phase φ[19,20]

$$\dot{u}_j=\mu\big(\upsilon_j\xi+F_j\cos\varphi\big),\quad \dot{\upsilon}_j=-\mu\big(u_j\xi+F_j\sin\varphi\big);\quad j=1,2$$
$$\dot{\xi}=\mu F_3,\quad \dot{\varphi}=1+\mu\xi \tag{1.4}$$

2. The averaged system

Averaging system (1.4) with respect to the rapidly rotating phase, we obtain the averaged system

$$\dot{u}_1=\mu\left(-\frac{h+h_1}{2}u_1-\frac{h_1}{2}\upsilon_2+\upsilon_1\xi+\frac{\nu}{2}\right),\quad \dot{\upsilon}_1=\mu\left(-\frac{h+h_1}{2}\upsilon_1+\frac{h_1}{2}u_2-u_1\xi\right)$$
$$\dot{u}_2=\mu\left(-\frac{h+h_1}{2}u_2+\frac{h_1}{2}\upsilon_1+\upsilon_2\xi\right),\quad \dot{\upsilon}_2=\mu\left(-\frac{h+h_1}{2}\upsilon_2-\frac{h_1}{2}u_1-u_2\xi-\frac{\nu}{2}\right)$$
$$\dot{\xi}=\mu\left(\Delta-\gamma\xi-\frac{\lambda}{2}(u_1-\upsilon_2)-\frac{\chi}{2}\Big[(u_1+\upsilon_2)^2+(\upsilon_1-u_2)^2\Big]\right),\quad \dot{\varphi}=1+\mu\xi \tag{2.1}$$

In system (2.1) the former notation is retained for the averaged variables. The subsystem of the first five equations of the system has been separated from the sixth and can be investigated separately. This subsystem has a property that greatly simplifies the problem. We will formulate this property in the form of the following assertion.

In the phase space $G(u_1, u_2, \upsilon_1, \upsilon_2, \xi)$ of system (2.1) there is a single, globally asymptotically stable invariant manifold $M=\{u_1=-\upsilon_2, \upsilon_1=u_2\}$.

Proof. In system (2.1) we introduce variables of the form

$$u_1+\upsilon_2=x,\quad \upsilon_1-u_2=y$$

(after such a replacement, the investigation of the properties of the manifold reduces to an investigation of the properties of the equilibrium state $u_1+\upsilon_2=x=0$, $\upsilon_1-u_2=y=0$). As a result we obtain a system of the form

$$\dot{x}=-\frac{h+2h_1}{2}x+y\xi,\quad \dot{y}=-\frac{h+2h_1}{2}y-x\xi \tag{2.2}$$

The equilibrium state $x=0$, $y=0$ of system (2.2) is clearly unique. On the other hand, by virtue of this system, the derivative of Lyapunov's function $V=x^2+y^2$ is

$$\dot{V}=-(h+2h_1)(x^2+y^2)\le 0,\quad \forall(u_1,u_2,\upsilon_1,\upsilon_2,\xi)\in G$$

which proves the assertion.

The manifold $u_1=-\upsilon_2=u$, $\upsilon_1=u_2=\upsilon$ is filled by phase trajectories of the three-dimensional dynamical system

$$\dot{u}=\mu\left(-\frac{h}{2}u+\upsilon\xi+\frac{\nu}{2}\right),\quad \dot{\upsilon}=\mu\left(-\frac{h}{2}\upsilon-u\xi\right),\quad \dot{\xi}=\mu(\Delta-\gamma\xi-\lambda u) \tag{2.3}$$

The dynamics of system (2.1) is completely specified by the dynamical properties of system (2.3).

3. The rotation characteristic of the motor. The resonance characteristic of the shaft vibrations

Definition. We will call the function

$$\Omega=\lim_{T^*\to\infty}\frac{1}{T^*}\int_0^{T^*}\dot{\varphi}(t,t_0)dt$$

which is specified over the parameter space of system (1.1) and the space of its initial conditions, the rotation characteristic of the motor.

The equality

$$\Omega = \lim_{T^* \to \infty} \frac{1}{T^*} \int_0^{T^*} \dot{\varphi}^*(t, t_0) dt$$

where $\dot{\varphi}^*(t, t_0)$ is one of the limit solutions of system (1.1), holds.

In fact, the first solution $\dot{\varphi}(t, t_0)$ can be represented in the form $\dot{\varphi}(t, t_0) = \tilde{\dot{\varphi}}(t, t_0) + \dot{\varphi}^*(t, t_0)$, where $\tilde{\dot{\varphi}}(t, t_0)$ is the transitional process to the limit solution $\dot{\varphi}^*(t, t_0)$, i.e.,

$$\lim_{T^* \to \infty} \dot{\varphi}(t, t_0) = \dot{\varphi}^*(t, t_0), \quad \lim_{T^* \to \infty} \tilde{\dot{\varphi}}(t, t_0) = 0$$

By definition

$$\Omega = \lim_{T^* \to \infty} \frac{1}{T^*} \int_0^{T^*} \tilde{\dot{\varphi}}(t, t_0) dt + \lim_{T^* \to \infty} \frac{1}{T^*} \int_0^{T^*} \dot{\varphi}^*(t, t_0) dt$$

For simplicity, we will assume that the stability of the limit solution is exponential. we than have

$$\left\| \tilde{\dot{\varphi}}(t, t_0) \right\| < D\exp(-\lambda t)$$

$$\lim_{T^* \to \infty} \frac{1}{T^*} \int_0^{T^*} \tilde{\dot{\varphi}}(t, t_0) dt < \lim_{T^* \to \infty} \frac{1}{T^*} \int_0^{T^*} \left\| \tilde{\dot{\varphi}}(t, t_0) \right\| dt < \lim_{T^* \to \infty} \frac{1}{T^*} \int_0^{T^*} D\exp(-\lambda t) dt = 0$$

which proves the above assertion.

In particular, if $\dot{\varphi}^*(t, t_0)$ is a periodic solution, then

$$\dot{\varphi}^*(t, t_0) = \Omega(.,\gamma) + R(t)$$

where $R(t)$ is a Fourier series with zero mean. As a result, we obtain $\Omega = \Omega(., \gamma)$.

By definition, as well as by virtue of the averaging principle and the last equation in system (2.1), we find that $\Omega = 1 + \mu\langle\xi^*(t, t_0)\rangle_t$, where $\xi^*(t, t_0)$ is the solution corresponding to a certain limit set of system (2.3), which is obtained under assigned initial conditions (the solutions corresponding to transitional processes give zero values when the limit is calculated). Thus, the investigation of the rotation characteristics involves a non-local investigation of system (1.1) and of system (2.3) in the zone of the fundamental resonance. In other words, the construction of qualitatively different rotation characteristics requires the solution of the classical oscillation-theory problem of partitioning the parameter space into regions that correspond to different structures of the phase trajectories of the system. We recall the relation between the limit sets of averaged and original systems.[21] The equilibrium state of an averaged system is the limit cycle of the original system, the limit cycle is a two-dimensional torus, etc.

For practical purposes the behavior of the rotation characteristic as a function of the constant component of the torque of the motor (T in equalities (1.1)) is of greatest interest. In the case considered here, it is more convenient to plot and analyse the inverse dependence: $T = T(\Omega)$. Since T and Δ, as well as Ω and $\langle\xi\rangle_t$, are linearly related, it is convenient to plot the function $\Delta = f(\langle\xi\rangle_t)$. We will use this function as the rotation characteristic, since it contains all the qualitative features of the true rotation characteristic. Moreover, a simple conversion transforms one into the other.

Definition. We will call the function $A(\Omega) = \max_t \sqrt{x_*^2 + y_*^2}$, which is specified over the parameter space of system (1.1) and the space of its initial conditions, the resonance characteristic of the flexible rotor. Here $x_* = x_*(t, t_0)$ and $y_* = y_*(t, t_0)$ specify the solution of system (1.1) corresponding to the limit set of its phase trajectories that is realized under the assigned initial conditions.

The resonance characteristic specifies the maximum deviation of the flexible rotor from the unperturbed position as a function of the rotation frequency of the motor. In particular, if the motor has an unlimited power and its rotation frequency does not depend on the load, the resonance characteristic is the amplitude–frequency characteristic.

Like the rotation characteristic, we will plot the resonance characteristic in the resonance zone in the form $A = A(\Delta)$. The two characteristics are interrelated, since $A = A(\Delta)$ is the resonance characteristic and $\Delta = \Delta(\Omega)$ is the rotation characteristic. The geometrical form of the resonance characteristic and the stability of its various branches are determined by the form and stability of the rotation characteristic.

4. Properties of the averaged system, the rotation characteristic and the resonance characteristic

In system (2.3) we make the time transformation $\mu\tau = \tau_n$ and introduce the following convenient notation: $\xi = x$, $u = y$ and $\upsilon = z$. As a result, instead of system (2.3), we obtain the equivalent system

$$\dot{x} = -\gamma x - \lambda y + \Delta, \quad \dot{y} = -\frac{h}{2} y + xz + \frac{\nu}{2}, \quad \dot{z} = -\frac{h}{2} z - xy \tag{4.1}$$

Property 1. All solutions of system (4.1) are bounded.

Proof. By virtue of system (4.1), the derivative of the quadratic form

$$V=\frac{x^2}{2}+\frac{y^2}{2}+\frac{(z-\lambda)^2}{2}$$

is

$$\dot{V}=-\gamma x^2-\frac{h}{2}y^2-\frac{h}{2}z^2+\Delta x+\frac{\nu}{2}y+\frac{\lambda h}{2}z$$

which is clearly negative outside a certain sphere R^2. This means that in the phase space $G(x, y, z)$ there is a sphere $V \le C^2$ that absorbs the sphere R^2, into which all the phase trajectories of system (4.1) enter and remain. In other words,

$$||x||,\ ||y||,\ ||z|| < C < \infty \text{ at } t \to \infty$$

Property 2. Depending on the values of the parameters, system (4.1) has from one to three equilibrium states, whose coordinates are solutions of the equation

$$f=\Delta=\gamma\omega+\frac{\lambda\nu h/4}{\omega^2+h^2/4} \tag{4.2}$$

Here $x_0 = \omega$, where x_0 is the coordinate of the equilibrium state.

Note that curve (4.2) is the rotation characteristic corresponding to the equilibrium states of system (4.1) (the limit cycles of system (1.1)). The resonance characteristic corresponding to the equilibrium states has the form

$$A(\Delta)=\frac{\nu/2}{\sqrt{(\omega(\Delta))^2+h^2/4}} \tag{4.3}$$

It is not difficult to show that the rotation characteristic has two extrema (and three equilibrium states for values of Δ between these extrema) when the inequality

$$\frac{\gamma h^2}{\lambda\nu}<\frac{3\sqrt{3}}{4}$$

holds and that the rotation characteristics are one-to-one functions (there is one equilibrium state for all Δ) when the opposite inequality holds.

Property 3. If system (4.1) has a single equilibrium state for all values of Δ, it is globally asymptotically stable.

Proof. We will assume that $O(x_0, y_0, z_0)$ is an equilibrium state of system (4.1). We make the replacement

$$x=x_0+u,\quad y=y_0+\upsilon,\quad z=z_0+w$$

as a result of which the system takes the form

$$\dot{u}=-\gamma u-\lambda\upsilon,\ \ \dot{\upsilon}=-\frac{h}{2}\upsilon+x_0w+z_0u+uw,\ \ \dot{w}=-\frac{h}{2}w-x_0\upsilon-y_0u-u\upsilon \tag{4.4}$$

By virtue of system (4.4), the derivative of Lyapunov's function $V=\frac{1}{2}(mu^2+\upsilon^2+w^2)$ is non-positive:

$$\dot{V}=-\frac{h}{2}(\alpha_1u+\upsilon)^2-\frac{h}{2}(\alpha_2u+w)^2\le 0,\quad \forall(u,\upsilon,w)\in G;\quad \alpha_1=\frac{m\lambda-z_0}{h},\quad \alpha_2=\frac{y_0}{h}$$

where m is a positive root of the equation

$$\lambda^2m^2-2(\lambda z_0+h\gamma)m+y_0^2+z_0^2=0$$

The condition for a positive root to exist is identical to the condition for a one-to-one rotation characteristic.

The one-to-one rotation characteristics in Fig. 2*a* correspond to the single-valued resonance characteristics in Fig. 3*a*.

Property 4. In the case of rotation characteristics that are not one-to-one, the characteristic equation of the system in variations relative to an arbitrary equilibrium state $O(x_0, y_0, z_0)$ has the form

$$p^3+a_0p^2+a_1p+a_2=0$$

$$a_0=h+\gamma,\quad a_1=\chi-\frac{\lambda\nu\omega}{2\chi}+\gamma h,\quad a_2=\gamma\chi-\frac{\lambda\upsilon h\omega}{2\chi},\quad \chi=\frac{h^2}{4}+\omega^2$$

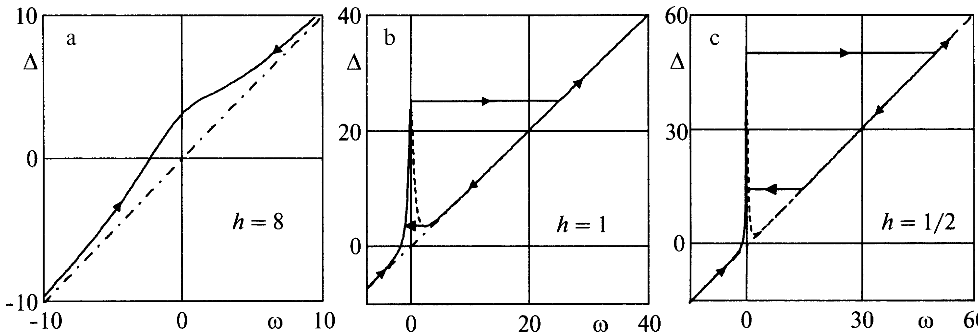

Fig. 2.

The Hurwitz stability conditions are expressed by the inequalities

$$a_1 = f_1(\omega) = \chi + \gamma h - \frac{\lambda\nu\omega}{2\chi} > 0, \quad a_2 = f_2(\omega) = \gamma\chi - \frac{\lambda\nu h\omega}{2\chi} = \chi\Delta'_\omega > 0$$

$$a_0 a_1 - a_2 = f_3(\omega) = h\chi + \gamma h(h+\gamma) - \frac{\lambda\nu\gamma\omega}{2\chi} > 0 \tag{4.5}$$

It follows from the second inequality in (4.5) that the descending segment of the rotation characteristic ($\Delta'_\omega < 0 \Rightarrow a_2 < 0$) is unstable for any parameter values. Each point on it corresponds to a saddle equilibrium state.[22]

Property 5. When $h \geq \gamma$, both ascending segments of the rotation characteristic are stable at all their points.

Proof. In fact, since $\Delta'_\omega > 0$, we have $a_2 > 0$, and the inequality

$$\gamma\chi > \frac{\lambda\nu h\nu}{2\chi}$$

holds, by virtue of which we obtain

$$a_1 = f_1(\omega) = \chi + \gamma h - \frac{\lambda\nu\omega}{2\chi} > \chi + \gamma h - \frac{\gamma\chi}{h} = \frac{(h-\gamma)\chi}{h} + \gamma h > 0$$

$$a_0 a_1 - a_2 = f_3(\omega) > h\chi + \gamma h(h+\gamma) - \frac{\gamma^2\chi}{h} = \frac{\left(h^2-\gamma^2\right)\chi}{h} + \gamma h(h+\gamma) > 0$$

Thus, the stability conditions hold for all points on the ascending segments of the rotation characteristic when $h \geq \gamma$.

In this case the equilibrium states corresponding to points on the ascending segments are stable foci or nodes. At the extrema on the rotation characteristics, a merging bifurcation of the equilibrium states with the formation of a saddle-node followed by its disappearance after an appropriate change in the parameter Δ occurs.

Property 6. Note that the stability conditions hold for all points on the rotation characteristic at $\omega < 0$. Note also that the stability conditions, i.e., the first and third inequalities in (4.5), become stronger as h is increased ($\partial f_1/\partial h > 0$, $\partial f/\partial h > 0$) (this corresponds to the physical meaning). We will assume that $h \ll 1$. In this case, the y coordinates of the left-hand (maximum) and right-hand (minimum) extrema on the rotation characteristic are specified, respectively, by the asymptotic formulae

$$\omega_1 = \frac{h^3\gamma}{8\lambda\nu}, \quad \omega_2 = \left(\frac{h\lambda\nu}{2\gamma}\right)^{1/3}$$

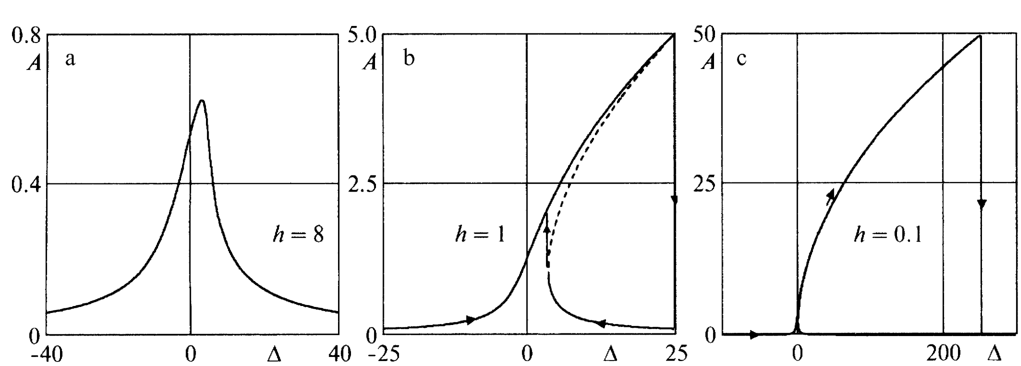

Fig. 3.

For $\omega < \omega_1$ we obtain the asymptotic expressions

$$f_1 = \gamma h - \frac{2\lambda\nu\omega}{h^2} > 0 \Rightarrow \omega < \frac{\gamma h^3}{2\lambda\nu} = 4\omega_1$$
$$f_3 = \gamma^2 h - \frac{2\lambda\nu\gamma\omega}{h^2} > 0 \Rightarrow \omega < \frac{\gamma h^3}{2\lambda\nu} = 4\omega_1$$

Thus, when $h \ll 1$, the stability conditions hold for all points on the left-hand ascending segment of the rotation characteristic. By virtue of the monotonic increase in the functions f_1 and f_3 with h, these conditions hold for any h.

For points at $\omega > \omega_2$ on the right-hand ascending segment we obtain the asymptotic formulae

$$f_1 = \omega^2 - \frac{\lambda\nu}{2\omega} > 0 \Rightarrow \omega > \left(\frac{\lambda\nu}{2}\right)^{1/3} = \omega_3$$
$$f_3 = \gamma^2 h - \frac{\lambda\nu\gamma}{2\omega} > 0 \Rightarrow \omega > \frac{\lambda\nu}{2\gamma h} = \omega_4$$

As ω varies from high to low values, the equilibrium state on the ascending segment loses stability, becomes a saddle-focus and remains such in the interval $\omega_3 < \omega < \omega_4$. A loss of stability occurs because the unstable limit cycle born from the separatrix loop of the saddle equilibrium state becomes trapped in the equilibrium state. Then the equilibrium state becomes a saddle[22] and remains such in the interval $\omega_2 < \omega < \omega_3$ until it vanishes at the minimum point of the rotation characteristic. Because the functions f_1 and f_2 are monotonic, there is a value $h = h_0 < \gamma$ such that part of the right-hand segment of the rotation characteristic is unstable in the interval $h < h_0$.

5. Qualitatively different patterns of rotation and resonance characteristics. The dynamics of the system as it passes through resonance for different viscosity coefficients of the medium

Based on the properties of the averaged system, we will present qualitatively different patterns of the rotation and resonance characteristics, as well as we will describe the corresponding behaviour of a flexible-shaft/motor system when quasistatic variation of the torque and passage through the resonance zone occurs. We will assume that the system parameters λ, ν and γ are fixed, but the viscosity coefficient of the medium h can vary from case to case. In the examples presented below $\lambda = \nu = 5$, and $\gamma = 1$.

In the case when $h > \sqrt{3\sqrt{3}\lambda\nu/(4\gamma)}$, the rotation characteristics are one-to-one. An example of such a rotation characteristic is shown in Fig. 2*a*. The resonance characteristic is a single-valued function (Fig. 3*a*). At high viscosities of the medium, the resonance properties of the shaft are weakly displayed, and passage through the resonance zone occurs in the left-to-right direction and in the opposite direction along the same path without jumps in the rotation frequency of the motor.

In the case when $h_0 \le h < \sqrt{3\sqrt{3}\lambda\nu/(4\gamma)}$, examples of the rotation characteristic and the corresponding resonance characteristic for parameters from this region are shown in Figs 2*b* and 3*b*. In this case the rotation characteristic has two extrema. The descending segment of the rotation characteristic is unstable, and both ascending segments are stable. As the torque of the motor increases, i.e., as the motor speeds up (the arrows pointing to the right), the rotation frequency of the motor becomes "stuck" at a constant value as the resonance frequency of the shaft is approached (the energy imparted to the motor is expended on enhancing the vibrations of the shaft). When the torque reaches the value corresponding to the left-hand extremum on the rotation characteristic, the rotation frequency of the motor jumps to the point on the right-hand branch that corresponds to a stable rotation mode. When the torque varies in the opposite direction, i.e., when the motor slows down (the arrows pointing to the right), the jump in the rotation frequency of the motor occurs from the left-hand extremum on the rotation characteristic to a stable point on the right-hand branch. The corresponding motion along the resonance characteristic is indicated by the arrows in Fig. 3*b*.

Examples of rotation and resonance characteristics corresponding to the case when $h < h_0$ are shown in Figs 2*c* and 3*c*. The phenomenon of a rotation frequency stuck at a constant value near resonance is intensified. In addition, when the motor slows down, the right-to-left jump in the rotation frequency occurs from the point $\omega = \omega_4$ on the right-hand ascending branch of the rotation characteristic, rather than from the minimum. The resonance characteristic becomes narrow, so that its ascending and descending segments practically merge.

We will estimate the size of the jump in the rotation frequency of the motor and the amplitude of the flexural vibrations in the case of small h. The left-to-right frequency jump occurs from the extremum on the rotation characteristic, from a point at $\omega \approx 0$. In this case $\Delta(0) = \lambda\nu/h$. The jump occurs to a point on the rotation characteristic that lies nearly on the asymptote $\Delta = \gamma\omega$ at $\omega = \lambda\nu/(\gamma h)$, i.e., $\delta\omega = \lambda\nu/(\gamma h)$. From the formulae that relate the dimensional and dimensionless parameters, we obtain $\delta\Omega = c^2e^2/(\varepsilon q\omega_0)$. For small values of ε, the size of the jump is considerable.

It is clear that

$$\max A(\Delta) = A(\Delta_0) = \nu/h \text{ when } \Delta_0 = \lambda\nu/h$$

In dimensional quantities we have

$$\max A(T/q) = A(\Omega_0) = ec/(\varepsilon\omega_0) \text{ when } \Omega_0 = \omega_0 + c^2e^2/(\varepsilon q\omega_0)$$

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